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R-separable solutions of Einstein's field equations

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Abstract. R-separable coordinate systems are introduced as a new class of systems in which the equations of general relativity may be solved. The static, axially symmetric vacuum and electrovac Einstein field equations are solved for two such systems, tangent spheres and bispherical. The bispherical is found to be more versatile than any previously used coordinate system in that its eigensolutions can represent the exteriors of single point, double point and line sources. The tangent sphere eigensolutions are found to be generalisations of the Curzon solution. The relatively simple nature of the individual bispherical eigensolutions allows explicit integration of the field equations for a completely general, static, two-body source. The bodies are then charged, according to the Weyl formalism, and the conditions for balance obtained. Finally, it is shown that all vacuum Weyl solutions are either type I or type D.

1. Introduction

In adapting any mathematical model to represent a particular realistic physical situation, the question of which coordinates to employ will inevitably arise. Obviously in a generally covariant theory, like the general theory of relativity, it is immaterial, from a purely formal point of view, which coordinates are ultimately chosen. However from a practical point, a judicious choice of coordinate systems can make the difference between solving the problem completely (mathematically exact) and not obtaining any solution whatsoever. Indeed, even though the idea of fitting the coordinate system to the problem is not new and is frequently used in physics, the general relativist, more than most, understands the true significance and beauty of well chosen coordinates. He is most often plagued with a large number of unknowns which are generally coupled in a nonlinear manner.

As an outstanding example of (mathematically) well chosen coordinates, one can consider the Weyl (1917) form of the metric for a static, axially symmetric, vacuum space–time. It was shown by Weyl that the line element for all such systems is reducible, with the right choice ('cylindrical' type) of coordinates, to

$$ds^2 = e^{w}\,dt^2 - e^{v-w}(d\rho^2 + dz^2) - \rho^2\,e^{-w}\,d\varphi^2 \tag{1.1}$$

where w and v are functions of ρ and z. Moreover the Einstein vacuum field equations, $R_{\mu\nu} = 0$, reduce to

$$\nabla^2 w \equiv w_{\rho\rho} + w_{zz} + \rho^{-1} w_\rho = 0 \tag{1.2}$$

and

$$v_\rho = \tfrac{1}{2}\rho(w_\rho^2 - w_z^2), \qquad v_z = \rho w_\rho w_z. \tag{1.3}$$

Subscripts refer to partial differentiation with respect to the indicated variable. Consequently one need search only for harmonic functions.

Having established the Weyl coordinates and having tentatively regarded them as cylindrical (they are actually the analogue to the cylindrical coordinates of flat space), it is then natural, in the quest for solutions to the Laplace equation, to go one step further, guided by the symmetries of the source configuration, and consider other coordinates (oblate or prolate spheroidal) which are related in the usual way to the cylindrical coordinates of flat space. Many authors have used this procedure not only to find new exact solutions but also to help clarify and interpret known ones (Erez and Rosen 1959, Zipoy 1966, Bonnor and Sackfield 1968, Szekeres 1968, Voorhees 1970). The main purpose of this paper is to explore Weyl-type coordinate systems whose eigensolutions† can represent exteriors of bounded, physical sources such as 'multi-point' or 'bar' type.

Thus far only those simply separable coordinates have been used (Zipoy 1966, Bonnor and Sackfield 1968, Voorhees 1970) whose eigensolutions, for finite bounded systems, represent single body sources such as disc or single point (e.g. the Curzon solution in Weyl polar coordinates). Consequently a possible alternative would be the remaining simply separable coordinates based on first and second degree surfaces, of which there are eleven in total. An examination of these, however, reveals that the eigensolutions are not suitable. The next alternative, and the one that will be pursued here, is to investigate the more numerous R-separable coordinate systems (Moon and Spencer 1971). These afford a much richer variety‡.

The bispherical coordinate system (η, θ, φ), in particular, is found to be well suited for two-body configurations. This system has a remarkable versatility in that it offers coordinate separated (eigen) solutions for single point, double point, finite single line, infinite line and single and double semi-infinite line configurations. Previously obtained double point and finite line solutions§ have not been eigensolutions. The tangent-sphere coordinates (μ, ν, φ) yield a previously unobtained generalisation of the Curzon solution.

Here, only the vacuum solutions are studied. Consequently, the idealised mass distributions, as is commonly done (Zipoy 1966, Winicour *et al* 1968, Voorhees 1970, Cooperstock and Junevicus 1974), are inferred from singularities in the Riemann tensor (specifically the Kretchmann scalar).

2. Tangent sphere solution

In this coordinate system, the metric (1.1) becomes

$$ds^2 = e^w\, dt^2 - \frac{e^{v-w}}{(\mu^2+\nu^2)^2}(d\mu^2 + d\nu^2) - \frac{e^{-w}\mu^2}{(\mu^2+\nu^2)^2}(d\varphi)^2 \tag{2.1}$$

† That is, solutions of (1.2) and (1.3) which result when separable-variable solutions of the Laplace equation are used.

‡ In contrast to simply separable coordinates, there are theoretically an infinite number of R-separable coordinates.

§ Infinite superposition of eigensolutions.

where the coordinates are related by (see figure 1)

$$\mu = \rho/r^2 \equiv (\sin\theta)/r,$$
$$\nu = z/r^2 \equiv (\cos\theta)/r, \qquad r^2 = \rho^2 + z^2, \qquad 0 \leqslant \theta \leqslant \pi.$$

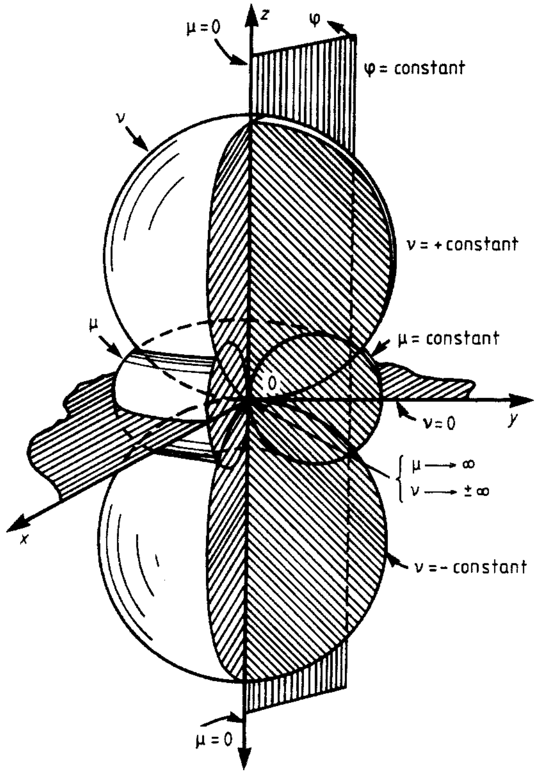

Figure 1. Tangent sphere coordinates.

With this metric, the vacuum field equations reduce to

$$\frac{\partial}{\partial\mu}\left(\frac{\mu w_\mu}{\mu^2+\nu^2}\right) + \mu\frac{\partial}{\partial\nu}\left(\frac{w_\nu}{\mu^2+\nu^2}\right) = 0, \tag{2.2}$$

$$v_\nu = [\mu/(\mu^2+\nu^2)][\mu\nu(w_\mu^2 - w_\nu^2) - (\mu^2-\nu^2)w_\nu w_\mu], \tag{2.3a}$$

$$-v_\mu = [\mu/2(\mu^2+\nu^2)][4\mu\nu w_\mu w_\nu + (\mu^2-\nu^2)(w_\mu^2 - w_\nu^2)]. \tag{2.3b}$$

By substituting

$$w = (\mu^2+\nu^2)^{1/2} M(\mu) N(\nu) \tag{2.4}$$

into (2.2), it then follows that M and N satisfy

$$\frac{\mathrm{d}^2 M}{\mathrm{d}\mu^2} + \frac{1}{\mu}\frac{\mathrm{d}M}{\mathrm{d}\mu} - kM = 0, \qquad \frac{\mathrm{d}^2 N}{\mathrm{d}\nu^2} + kN = 0, \tag{2.5}$$

k being the separation constant. Setting $k=-q^2$†, the solutions of (2.5) are as follows.

Class (i): $q \neq 0$.

$$M = AJ_0(q\mu)+BY_0(q\mu) \equiv M_0, \qquad N = C\,e^{q\nu}+D\,e^{-q\nu}. \tag{2.6}$$

Class (ii): $q=0$.

$$M = A+B\ln(\mu), \qquad N = C+D\nu. \tag{2.7}$$

A–D and q are arbitrary constants. J_0 and Y_0 are the zeroth-order Bessel functions of the first and second kind respectively.

Substituting (2.4) into equations (2.3) yields

$$\partial v/\partial\nu = -\mu^2M^2\dot{N}(N+\nu\dot{N})+\mu\nu\dot{M}N^2(M+\mu\dot{M})-\mu(\mu^2-\nu^2)M\dot{M}N\dot{N}, \tag{2.8a}$$

$$-\partial v/\partial\mu = \mu MN(N+2\nu\dot{N})(\tfrac{1}{2}M+\mu\dot{M})+\tfrac{1}{2}\mu(\mu^2-\nu^2)(N^2\dot{M}^2-M^2\dot{N}^2), \tag{2.8b}$$

where a dot denotes differentiation with respect to μ or ν. After a lengthy calculation, (2.8) yields the following.

Class (i).

$$\begin{aligned}{}^{\ddagger}v = &-\tfrac{1}{2}\alpha\mu[q\nu^2M_0M_1+\tfrac{1}{2}\mu(M_0^2+M_1^2-2q\mu M_0M_1)]+\tfrac{1}{2}q\beta\mu^2\nu(M_1^2-M_0^2)\\ &+q^2CD\mu^2(\nu^2-\tfrac{1}{3}\mu^2)(M_0^2+M_1^2-M_0M_1/q\mu)-\tfrac{1}{6}CD\mu^2(3M_0^2-M_1^2)+E_0\end{aligned} \tag{2.9}$$

where

$$M_1 \equiv AJ_1(q\mu)+BY_1(q\mu), \qquad \alpha \equiv C^2\,e^{2q\nu}+D^2\,e^{-2q\nu}, \qquad \beta \equiv C^2\,e^{2q\nu}-D^2\,e^{-2q\nu}. \tag{2.10}$$

J_1 and Y_1 are the first-order Bessel functions of the first and second kind respectively.

Class (ii).

$$\begin{aligned}v = {}&B\nu(B+M)\left[\frac{D^2\nu^3}{4}+\frac{2CD\nu^2}{3}+\frac{\nu}{2}\left(C^2-\frac{D^2\mu^2M}{B}\right)-\frac{CD\mu^2M}{B}\right]\\ &+D\nu^2M\left(\frac{BD\nu^2}{4}+\frac{BC\nu}{3}-\frac{D\mu^2M}{2}\right)+F(\mu)\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}F(\mu)={}&\frac{\mu^2}{4}\left[\frac{D^2\mu^2}{2}\left(A^2+\frac{B^2}{8}-\frac{AB}{2}\right)-C^2\left(A^2+\frac{B^2}{2}+AB\right)\right]-\tfrac{1}{4}BC^2\mu^2\ln(\mu)\\ &\times[2A+B+B\ln(\mu)]+\tfrac{1}{4}BD^2\mu^4\ln(\mu)[A-\tfrac{1}{4}B+\tfrac{1}{2}B\ln(\mu)]+E_1.\end{aligned} \tag{2.12}$$

E_0 and E_1 are arbitrary constants.

The function $Y_0(q\mu)$ is singular ($\to -\infty$) as $\mu\to 0$ whereas $J_0(q\mu)$ is regular everywhere. Consequently not only the metric, but also the Kretchmann scalar,

$$R^2 \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 8\,e^{2w-2v}[(R^1_{\ 212})^2+(R^3_{\ 131})^2+(R^0_{\ 101})^2+2(R^3_{\ 132})^2], \tag{2.13}$$

† The case $k=q^2$ can be recovered from the above solutions by letting $q\to iq$, $C\to\frac{1}{2}(C-iD)$, $D\to\frac{1}{2}(C+iD)$ in all expressions.

‡ $\int x^3(Z_0^2+Z_1^2)\,dx=\frac{1}{3}x^4(Z_0^2+Z_1^2-Z_0Z_1/x)+\frac{1}{3}x^2Z_1^2$, $\quad Z_0\equiv AJ_0(x)+BY_0(x)$, $Z_1\equiv AJ_1(x)+BY_1(x)$.

is singular† all along the z axis unless $B = 0$, in which case they are both singular only at the origin. Therefore it is reasonable to interpret the class (i) solution, for $B \neq 0$, as that describing the exterior of some idealised infinite line mass located along the z axis. For $B = 0$, the source, with total mass $m = -\frac{1}{2}A(C+D)$, is point-like and located at the origin. For the class (ii) solution one finds that the behaviour of the metric and scalar is essentially the same as in the class (i) case. Note that for $D \neq 0$, $B = 0$, the harmonic potential is $C/r + Dz/r^3$ and the solution corresponds to some generalisation of the Curzon one (Synge 1960).

3. Bispherical solution

In this coordinate system the metric (1.1) becomes

$$ds^2 = e^w\, dt^2 - \frac{a^2 e^{v-w}}{(\cosh\eta - \cos\theta)^2}(d\eta^2 + d\theta^2) - \frac{a^2 \sin^2\theta\, e^{-w}}{(\cosh\eta - \cos\theta)^2} d\varphi^2 \quad (3.1)$$

where the coordinates are related by (see figure 2)

$$\rho^2 + (z - a\coth\eta)^2 = a^2/\sinh^2\eta, \qquad r^2 - 2a\rho\cot\theta = a^2.$$

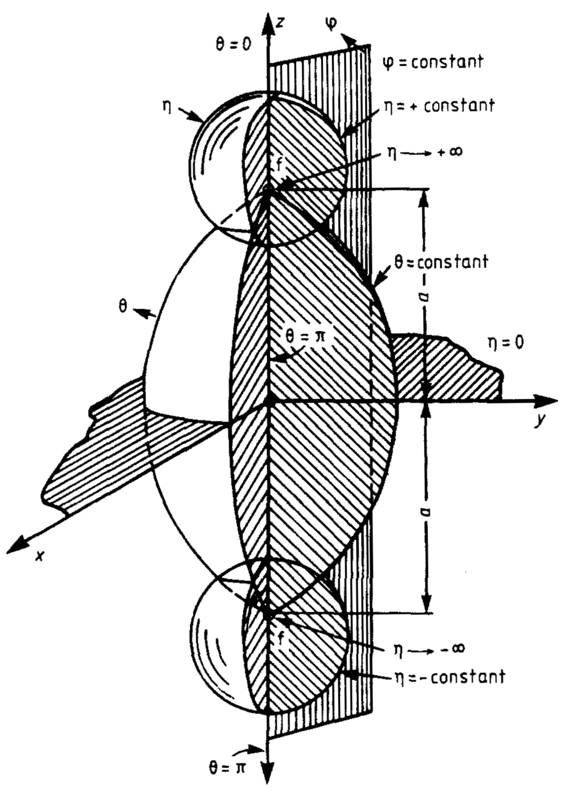

Figure 2. Bispherical coordinates.

† There exists at least one path of approach, to the point in question, in which R^2 becomes singular (Gautreau and Hoffman 1969, Cooperstock and Junevicus 1974).

The vacuum field equations reduce to

$$\sin\theta\frac{\partial}{\partial\eta}\left(\frac{w_\eta}{\cosh\eta-\cos\theta}\right)+\frac{\partial}{\partial\theta}\left(\frac{\sin\theta\, w_\theta}{\cosh\eta-\cos\theta}\right)=0, \tag{3.2}$$

$$\frac{\partial v}{\partial\eta}=\frac{\sin\theta}{2(\cosh\eta-\cos\theta)}\times[(w_\theta^2-w_\eta^2)\sin\theta\sinh\eta-2w_\theta w_\eta(1-\cos\theta\cosh\eta)], \tag{3.3a}$$

$$-\frac{\partial v}{\partial\theta}=\frac{\sin\theta}{2(\cosh\eta-\cos\theta)}\times[(w_\theta^2-w_\eta^2)(1-\cos\theta\cosh\eta)+2w_\theta w_\eta\sin\theta\sinh\eta]. \tag{3.3b}$$

Solutions are sought which are of the form

$$w=(\cosh\eta-\cos\theta)^{1/2}M(\eta)N(\theta). \tag{3.4}$$

After substituting (3.4) into (3.2) one finds that M and N are required to satisfy

$$\mathrm{d}^2M/\mathrm{d}\eta^2-[\tfrac{1}{4}+s(s+1)]M=0, \tag{3.5}$$

$$\mathrm{d}^2N/\mathrm{d}\theta^2+\cot\theta\,\mathrm{d}N/\mathrm{d}\theta+s(s+1)N=0, \tag{3.6}$$

where $s(s+1)$ has been chosen as the separation constant. The solutions of (3.5) and (3.6) are

$$M(\eta)=A\,\mathrm{e}^{\eta(s+1/2)}+B\mathrm{e}^{-\eta(s+1/2)}, \tag{3.7}$$

$$N(\theta)=CP_s(\cos\theta)+DQ_s(\cos\theta), \tag{3.8}$$

where P_s and Q_s are the Legendre functions (Gradshteyn and Ryzhik 1965, Macrobert 1967, Erdelyi *et al* 1954) of the first and second kind respectively and A–D are arbitrary constants.

With the substitution of (3.4) into (3.3a,b), it follows that

$$\partial v/\partial\eta=\tfrac{1}{2}\sin\theta[M^2N\dot N\sinh\eta\cos\theta-2M\dot MN\dot N(1-\cos\theta\cosh\eta)-M\dot MN^2\sin\theta\cosh\eta+\tfrac{1}{4}\sin\theta\sinh\eta(4M^2\dot N^2-4\dot M^2N^2-M^2N^2)], \tag{3.9a}$$

$$-\partial v/\partial\theta=\tfrac{1}{2}\sin\theta[\tfrac{1}{4}M^2N^2(1+\cos\theta\cosh\eta)+(M^2\dot N^2-\dot M^2N^2)(1-\cos\theta\cosh\eta)+M^2N\dot N\sin\theta\cosh\eta+N^2M\dot M\cos\theta\sinh\eta+2M\dot MN\dot N\sin\theta\sinh\eta] \tag{3.9b}$$

where a dot denotes differentiation with respect to η or θ.

After some calculation, v is determined to be as follows.

Case (i): $s\neq-\tfrac{1}{2}, 0$.

$$v=\tfrac{1}{2}\sin\theta\{\Lambda(\eta)\sin\theta[\dot N^2+s(s+1)N^2]-M\dot MN^2\sinh\eta\sin\theta+M^2N\dot N(\cos\theta\cosh\eta-1)\}+F(\theta) \tag{3.10}$$

where

$$\Lambda(\eta)\equiv\int M^2\sinh\eta\,\mathrm{d}\eta \tag{3.11}$$

and $F(\theta)$, the integrating function, can be written as

$$F(\theta)=-2AB(s+\tfrac{1}{2})^2\int_\xi^1 N^2(\xi)\,\mathrm{d}\xi+E_1, \qquad \xi\equiv\cos\theta. \tag{3.12}$$

Case (ii): $s=0$.

$$v=\tfrac{1}{4}[2\Lambda(\eta)D^2+N^2\sin^2\theta\cosh\eta(2AB-M^2)-2DM^2N(\cos\theta\cosh\eta-1)]+F(\theta) \tag{3.13}$$

where

$$2F=-AB\int_\xi^1 N^2(\xi)\,\mathrm{d}\xi+\tfrac{1}{4}(A^2+B^2)[2D\cos\theta(C+DQ_0)-C^2\cos^2\theta$$
$$+2D^2\ln|\sin\theta|+D\sin^2\theta\,Q_0(2C+DQ_0)]+E_1. \tag{3.14}$$

Case (iii): $s=-\frac{1}{2}$.

$$v=\tfrac{1}{2}\sin\theta\cosh\eta(\dot{N}^2\sin\theta+N\dot{N}\cos\theta-\tfrac{1}{4}N^2\sin\theta)-\tfrac{1}{2}N\dot{N}\sin\theta+E_1. \tag{3.15}$$

$M\equiv 1$ by choice and E_1 is an arbitrary constant.

Some knowledge of the possible (idealised) sources for the space–time can be obtained by studying the singular behaviour of w and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. Table 1 summarises the results of this investigation. It shows that the bispherical eigensolutions are indeed versatile; more so than eigensolutions of any previously considered coordinate system. Certainly with the flexibility as expressed by the arbitrary constants $A-D$ and $s(\geq -\frac{1}{2})$,

Table 1.

Case	A	B	C	D	s $s\geq-\frac{1}{2}$	Singular† regions of W and R^2	Possible source
1	Any Both not simultaneously zero	Any	Any With D/C	$\neq 0$ $\neq -(2/\pi)\tan(s\pi)$	Any	z axis	Infinite line
2	As in case 1		0	$\neq 0$	$s=\frac{1}{2}$ odd integer	$z\geq a$ $z\leq -a$	Semi-infinite lines
3	As in case 1‡		$E\cos s\pi$	$-(2E/\pi)\sin s\pi$	$\neq n$	$z\geq a$ $z\leq -a$	Semi-infinite lines
4	As in case 1§		$\neq 0$	0	$\neq n$	$-a\leq z\leq a$	Finite line (bar)
5	$\neq 0$	$\neq 0$	$\neq 0$	0	n	$z=\pm a$	Two point source
6	$\neq 0$	0	$\neq 0$	0	n	$z=a$	Single point source
7	0	$\neq 0$	$\neq 0$	0	n	$z=-a$	Single point source

† $W\to\pm\infty$. Also see footnote on p 2404.
‡ $P_s(-\xi)=\cos(s\pi)P_s(\xi)-(2/\pi)\sin(s\pi)Q_s(\xi)$.
§ $s=-\frac{1}{2}$ corresponds to a 'pure' bar.

the individual eigensolutions† encompass a myriad of possible representations of exterior space–times corresponding to sources ranging from lines (finite and infinite; cases (1)–(4)) to point sources (single and double; cases (5)–(7)).

When $D=0$, the metric is asymptotically flat with total mass

$$m=-a(A+B)/\sqrt{2}. \tag{3.16}$$

Equation (3.16) in conjunction with the behaviour $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ suggests that for cases (5)–(7) the total mass, in a Newtonian sense, associated at each point $z=\pm a$, is $m_1=-aA/\sqrt{2}$ and $m_2=-aB/\sqrt{2}$ respectively‡.

In the next section a study is made of the geodesic equations in order to explore further the nature of the arbitrary parameters appearing in the above solution.

4. Geodesic 'force'

Consider the spatial parts of the geodesic equation, in the (ρ, z) coordinate system, applied to a test particle which is initially at rest. With the four-velocity as $u^\alpha = e^{-w/2}(1,0,0,0)$, the geodesic equation reduces to

$$\frac{d^2x^i}{ds^2}+\tfrac{1}{2}e^{2w-v}\frac{\partial w}{\partial x^i}(u^0)^2=0, \qquad x^\mu \equiv (t,\rho,z,\varphi), \qquad i=1,2. \tag{4.1}$$

The analysis for the different cases (see table 1) will be divided into two broad categories: (A) the infinite and semi-infinite lines, cases (1)–(3); (B) the finite sources, cases (4)–(7).

A. For these solutions, it is interesting to examine (4.1) for the ρ-component of force 'near' the z axis, i.e.

$$d^2\rho/ds^2+\tfrac{1}{2}e^{w-v}\partial w/\partial\rho=0. \tag{4.2}$$

As $\theta\to 0$,

$$w_\rho \to -(\cosh\eta-1)^{3/2}MD/a\theta. \tag{4.3}$$

Therefore acceleration is inward (for positive§ density line mass), in the region $\theta=0$ $(z>a, z<-a)$, providing $DM<0$. Similarly as $\theta\to\pi$ $(-a<z<a)$, the acceleration is inward if $KM>0$ but vanishes when $K=0$. This implies that $D=0$, $s=n$ or $C=0$, $s=\tfrac{1}{2}(2n+1)$ or

$$C=-\tfrac{1}{2}D\pi\cot(s\pi) \qquad (C,D\neq 0; s\neq n, \tfrac{1}{2}(n+1)) \qquad \text{where } n=0,1,2,\ldots.$$

These are precisely the conditions for producing the semi-infinite lines, cases (2) and (3), or the point sources, cases (5)–(7). Thus the strut produces no acceleration and consequently it can be concluded that it has zero mass density. This observation is consistent with the results of Israel (1977).

B. For these cases, it is more suitable to examine the geodesic equation in the (η,θ) coordinate system.

Now

$$d^2\hat{x}^i/ds^2-\tfrac{1}{2}\hat{g}^{ii}\partial w/\partial\hat{x}^i=0, \qquad \hat{x}^\alpha\equiv(t,\eta,\theta,\varphi) \qquad (\text{no sum on } i) \tag{4.4}$$

† And a superposition.

‡ It is worthwhile mentioning that this mass separation is also suggested by the geodesic equations.

§ In a Newtonian sense.

where the caret (ˆ) refers to quantities in the (η, θ) coordinate system. As $\theta \rightarrow \pi$,

$$w_\theta \rightarrow \frac{-2C}{\pi}(\cosh \eta + 1)^{1/2} \frac{M \sin(s\pi)}{\pi - \theta} \tag{4.5}$$

for the bar solutions. Therefore the θ component of acceleration is inward (positive mass) if $MC \sin(s\pi) > 0$ at the point in question. Similarly, for the particle solutions there is an attractive force at $z = \pm a$ $(\eta \rightarrow \pm\infty)$ if $ACP_s(\xi) < 0$ or $BCP_s(\xi) < 0$ respectively†. Thus it is evident that for $s \neq 0$ there are attractive–repulsive regions as θ varies between 0 and π. The special case $s = 0$ yields 'point' sources which are attractive for all θ (see figure 3).

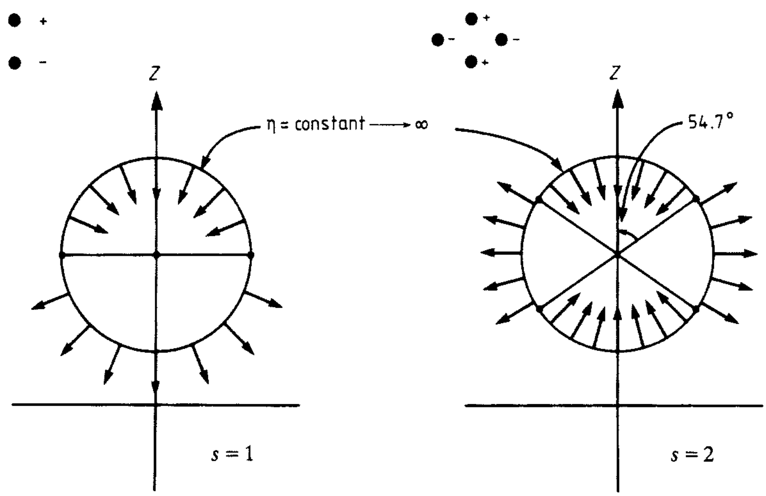

Figure 3. Force directions at $z = a$ with $AC < 0$.

It is clear from the above observations that the parameter s relates to the multipole structure of the particles (made up of positive and negative matter). Of course the actual multipole moments of the sources still remain obscure at this stage. A given s value does not explicitly yield that particular associated moment of the distribution (as it does in the classical case) but rather relates to a pseudo coordinate moment‡ correlated with the symmetrical placement of positive and negative mass 'unit' building blocks. The fact that the force, on a positive test particle, exhibits a change from negative to positive according to direction of approach indicates that the individual eigensolutions exhibit unphysical characteristics; but this is no different from the classical case of when a series solution is sought. In the classical case, physically meaningful solutions, consistent with the appropriate boundary conditions, are usually built up from a finite or infinite sum of the eigensolutions. Therefore, with this in mind, the next section will be devoted to the explicit determination of e^{υ} corresponding to a given sum of eigensolutions.

† Note, on the η = constant coordinate surfaces, η increases in the outward direction near $z = -a$ but inwardly near $z = +a$.

‡ The Schwarzschild solution appears to possess higher multipole structure in the Weyl coordinate system (Szekeres 1968).

5. General two-body solution

Motivated by the comments in the previous section, it would be interesting to consider a superposition, w, of eigensolutions which would correspond to a general two-point, axisymmetric, static source. The relevant eigensolutions would then be given by (3.4), (3.7) and (3.8) with $D \equiv 0$ and $s = 0, 1, 2, \ldots$. Therefore a suitable w will be of the form

$$w = \sum_{s=0}^{\infty} w_s = (\cosh \eta - \cos \theta)^{1/2} \sum_{s=0}^{\infty} M_s(\eta) N_s(\theta) \tag{5.1}$$

where

$$M_s \equiv A_s \, \mathrm{e}^{\eta(s+1/2)} + B_s \, \mathrm{e}^{-\eta(s+1/2)}, \qquad N_s \equiv P_s (\cos \theta). \tag{5.2}$$

After a lengthy computation, the field equations (1.3) determine v as

$$\begin{aligned} v = \frac{\sin \theta}{2} \sum_{p,s=0}^{\infty} \{&\sin \theta \, \Lambda_{ps} \{\dot{N}_p \dot{N}_s + N_p N_s [s(s+1) + \tfrac{1}{2}]\} - \sin \theta \, N_p N_s (M_p \dot{M}_s \sinh \eta \\ &+ \tfrac{1}{2} M_p M_s \cosh \eta - \Omega_{ps}) + \dot{N}_p N_s [\cos \theta (2\Omega_{ps} + \Lambda_{ps}) - 2\Pi_{ps}]\} + F(\theta) \end{aligned} \tag{5.3}$$

where

$$\Lambda_{ps} \equiv \int M_p M_s \cosh \eta \, \mathrm{d}\eta, \qquad \Pi_{ps} \equiv \int M_p \dot{M}_s \, \mathrm{d}\eta, \qquad \Omega_{ps} \equiv \int M_p \dot{M}_s \cosh \eta \, \mathrm{d}\eta$$

and $F(\theta)$ can be written as

$$\begin{aligned} F = &- \sum_{s=0}^{\infty} A_s B_s [2(s+\tfrac{1}{2})^2 I_s^s - (1-\xi^2) N_s \dot{N}_s] + \tfrac{1}{4} \sum_{s=0}^{\infty} (s+1)(A_{s+1} B_s + A_s B_{s+1}) \\ &\times \left(\frac{4(s+1)}{2s+1} [(s+1) I_{s+1}^{s+1} + s I_{s+1}^{s-1}] + \xi (N_{s+1}^2 - N_s^2) + I_{s+1}^{s+1} - I_s^s \right) \end{aligned} \tag{5.4}$$

with

$$I_s^p \equiv \int_{\xi}^{1} N_p N_s \, \mathrm{d}\xi.$$

The solution is asymptotically flat with total mass

$$m = \frac{-a}{\sqrt{2}} \left(\sum_{s=0}^{\infty} (A_s + B_s) \right). \tag{5.5}$$

As before (§ 3), the expression for m suggests that one associate the masses $m_1 = -(a/\sqrt{2}) \Sigma_{s=0}^{\infty} A_s$ and $m_2 = -(a/\sqrt{2}) \Sigma_{s=0}^{\infty} B_s$ at the singularities $z = a$ and $z = -a$ (on the z axis), respectively.

One immediate question that comes to mind, over sources of this nature, is the requirement for equilibrium. Such idealisations as struts are present to preserve the static nature of the space–time. But it can be shown that balanced (no strut) bodies can exist with the appropriate introduction of negative matter (Szekeres 1968). A more realistic situation might require electrostatic repulsion to overcome the gravitational attraction. To this end, in the final section of this paper, the point sources will be charged following the Weyl procedure ($g_{00}=g_{00}(\Phi)$)† (Majumdar 1947, Cooperstock and de la Cruz 1979). By then requiring equilibrium without a strut, the general condition for balance can be found within this framework.

6. Balance

If e^w and e^v in (1.1) constitute a solution to the vacuum field equations (1.2)–(1.3), then a Weyl (1917) solution ($e^{\bar{w}}$, $e^{\bar{v}}$, Φ) of the electrovac equations can be readily obtained by letting (Cooperstock and de la Cruz 1979)

$$e^w \to (1-b^2)^2 e^w/(b^2 e^w - 1)^2 \equiv e^{\bar{w}}, \qquad e^v \to e^v \equiv e^{\bar{v}} \tag{6.1}$$

and

$$\Phi = b(e^w - 1)/(b^2 e^w - 1) \tag{6.2}$$

where b is defined by

$$(1+b^2)/b = 2m/q. \tag{6.3}$$

The charged version, $(\bar{w}_s, \bar{v}_s, \Phi_s)$, of an individual vacuum eigensolution (w_s, v_s) can be easily obtained by appropriate use of equations (6.1)–(6.3) in conjunction with the solution presented in the previous section but with no sum‡.

As $r \to \infty$,

$$e^{\bar{w}_s} = 1 - \sqrt{2}a(b^2+1)(A_s+B_s)/(b^2-1)r + \ldots, \tag{6.4}$$

$$\Phi_s = \sqrt{2}ab(A_s+B_s)/(b^2-1)r + \ldots, \qquad \bar{v}_s \to 0. \tag{6.5}$$

Therefore, as before, associate masses and charges by

$$m_1 = \frac{aA_s}{\sqrt{2}}\left(\frac{b^2+1}{b^2-1}\right), \qquad m_2 = \frac{aB_s}{\sqrt{2}}\left(\frac{b^2+1}{b^2-1}\right), \tag{6.6}$$

$$q_1 = \frac{\sqrt{2}abA_s}{(b^2-1)}, \qquad q_2 = \frac{\sqrt{2}abB_s}{(b^2-1)}. \tag{6.7}$$

From (5.3)–(5.4),

$$\bar{v}_s(\rho = 0, -a < z < a) = -A_s B_s(2s+1) = -(2/a^2)(2s+1)(m_1 m_2 - q_1 q_2). \tag{6.8}$$

Thus the balance situation which is characterised by the absence of the strut ($\bar{v}_s = 0$) occurs when

$$m_1 m_2 = q_1 q_2 \tag{6.9}$$

which is the usual Newtonian balance condition.

From equations (6.6)–(6.7),

$$q_1/m_1 = q_2/m_2 \tag{6.10}$$

which together with (6.9) yields $m_1^2 = q_1^2$, $m_2^2 = q_2^2$. Thus for *every* individual eigensolution, balance is achieved only when the bodies are critically charged.

† Φ denotes the resulting electrostatic potential.

‡ Alternatively, the solution in § 3 with $D = 0$ and $s = 0, 1, 2, \ldots$.

Consider now the charged version of the general two-body solution, $w=\Sigma_{s=0}^{\infty} w_s$. Proceeding as before, one finds that the charge and mass separation are

$$m_1=\frac{a}{\sqrt{2}}\left(\frac{b^2+1}{b^2-1}\right)\left(\sum_{s=0}^{\infty} A_s\right), \qquad m_2=\frac{a}{\sqrt{2}}\left(\frac{b^2+1}{b^2-1}\right)\left(\sum_{s=0}^{\infty} B_s\right) \tag{6.11}$$

and

$$q_1=\frac{\sqrt{2}ab}{b^2-1}\left(\sum_{s=0}^{\infty} A_s\right), \qquad q_2=\frac{\sqrt{2}ab}{b^2-1}\left(\sum_{s=0}^{\infty} B_s\right). \tag{6.12}$$

Thus (6.10) holds generally for two charged bodies in the Weyl formalism (Gautreau *et al* 1972). However in this case, the more complicated condition

$$-\sum_{s=0}^{\infty}(2s+1)A_sB_s+\sum_{s=0}^{\infty}(s+1)[2(s+1)^2-1]\frac{(A_{s+1}B_s+A_sB_{s+1})}{(2s+1)(2s+3)}=0 \tag{6.13}$$

is required for balance. It is obvious from (6.13) that in general the Newtonian balance condition, for a superposition, will *not* hold. These results clearly indicate the nonlinear character of the field resulting from superposition. Linear superposition would have required

$$\left(\sum_{s=0}^{\infty} A_s\right)\left(\sum_{s=0}^{\infty} B_s\right)=0.$$

The above observations suggest that even in the Weyl formalism, it may be impossible for two spherically symmetric charged particles—if indeed one could recognise them as such (Bonnor 1981)—to be balanced solely with the Newtonian condition (6.9) (Barker and O'Connell 1977).

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Appendix

The Newman–Penrose method (Newman and Penrose 1962, Chandrasekhar 1979) will be used to determine the algebraic type. A suitable complex null basis, $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}})$, is

$$\boldsymbol{l}\equiv l^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-w/2}\frac{\partial}{\partial t}+\frac{\mathrm{e}^{w/2}}{\rho}\frac{\partial}{\partial\varphi}\right), \qquad \boldsymbol{n}\equiv n^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-w/2}\frac{\partial}{\partial t}-\frac{\mathrm{e}^{w/2}}{\rho}\frac{\partial}{\partial\varphi}\right),$$

$$\boldsymbol{m}\equiv m^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{\mathrm{e}^{(w-v)/2}}{\sqrt{2}}\left(\mathrm{i}\frac{\partial}{\partial\rho}+\frac{\partial}{\partial z}\right), \qquad \bar{\boldsymbol{m}}\equiv\text{complex conjugate of } \boldsymbol{m},$$

so that

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu$$

is the contravariant form of the Weyl metric (1.1).

The tetrad components of the Riemann tensor are

$$\Psi_0 \equiv -R_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta = \tfrac{1}{2} e^{-v}(R_{0101} - R_{0202} - 2iR_{0102}),$$

$$\Psi_1 \equiv -R_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta = 0,$$

$$\Psi_2 \equiv -R_{\alpha\beta\gamma\delta} l^\alpha n^\beta (l^\gamma n^\delta - m^\gamma \bar{m}^\delta) = -R_{0303}/2\rho^2,$$

$$\Psi_3 \equiv R_{\alpha\beta\gamma\delta} l^\alpha n^\beta n^\gamma \bar{m}^\delta = 0,$$

$$\Psi_4 \equiv -R_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta = \bar{\Psi}_0.$$

By considering the roots, ξ, of the equation

$$\Psi_0\xi^4 + 4\Psi_1\xi^3 + 6\Psi_2\xi^2 + 4\Psi_3\xi + \Psi_4 = 0$$

it follows that the Weyl metrics can only be of type I or type D. If $\Psi_0 = \Psi_4 = 0$ ($\Psi_2 \neq 0$) or $9\Psi_2^2 = \Psi_0\Psi_4$ then the space–time is Petrov type D, otherwise it is type I†.

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† Excluding the trivial case $\Psi_0 = \Psi_2 = \Psi_4 = 0$, type O.